ON THE STABILITY IN FIRST APPROXIMATION OF SYSTEMS WITH RANDOM LAG

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I.Ia. KATS
(Sverdlovsk)

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This paper is concerned with the stability of motion of a linear system with delay. The delay $\eta(t)$ is homogeneous to a Markov random process. Sufficient conditions of asymptotic stability are deduced from the probability of the unperturbed motion of that system. The problem is solved by the method of a Liapunov function [1] taking into consideration the circumstances due to the random character of the delay.

1. Let the equations of the disturbed motion have the form

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t - \eta(t)) \tag{1.1}$$

Here $x = \{x_1 \ldots, x_n\}$ is a n-dimensional vector of the phase coordinates of the system; A, B are constant $n \times n$ matrices; $\eta(t)$ is a purely discontinuous Markov random process [2] whereupon the quantity $\eta(t)$ can take values of the interval $[0, h]$, $h > 0$.

Let us assume that the statistical properties of the process $\eta(t)$ are given by the changing probability $P(t, \eta, \alpha)$, which has the Expansion ([2], p. 231)

$$P\{\eta(\tau) = \alpha,\ 0 \leqslant \tau \leqslant t \mid \eta(0) = \alpha\} = 1 - q(\alpha)\, t + o(t) \tag{1.2}$$

$$P\{\eta(t) \leqslant \beta,\ \eta(t) \neq \alpha \mid \eta(0) = \alpha\} = 1 - q(\alpha, \beta)\, t + o(t) \tag{1.3}$$

Here $P\{A|B\}$ is the conditional probability of the event A; $o(t)$ is an infinitely small number of an order of smallness larger than t when $t \to 0$.

The functions $q(\alpha)$, $q(\alpha, \beta)$ are known continuous functions of the parameter α, whereupon

$$q(\alpha, \beta) = q(\alpha) \quad \text{for } \beta \geqslant h, \qquad q(\alpha, \beta) = 0 \quad \text{for } \beta < 0$$
$$0 < q(\alpha) < q = \text{const} \tag{1.4}$$

We shall limit ourselves to the consideration of two cases: 1. The function $q(\alpha, \beta)$ has a continuous $p(\alpha, \beta)$

$$q(\alpha, \beta) = \int_0^\beta p(\alpha, \beta)\, d\beta$$

2. The function $\eta(t)$ can take a finite number of values $\{\eta_1, \ldots, \eta_r\} \in [0, h]$ whereupon the probability $p_{ij}(t)$ of the changes from a condition η_i into a condition η_j in the time t is determined by Eqs.

$$p_{ij}(t) = \alpha_{ij} t + o(t) \qquad (\alpha_{ij} = \text{const};\ i \neq j;\ i, j = 1, \ldots, r)$$

As it is known [2], under those assumptions it can be assumed that almost all the realizations $\eta(\omega, t)$ of the process $\eta(t)$ are step functions. We shall assume furthermore that they are continuous on the right.

Let the initial conditions at the instant $t = 0$, which determine the motion for $t \geqslant 0$, be given in the form of a segment of the trajectory $x_0(\vartheta_0)$ $(-h \leqslant \vartheta_0 \leqslant 0)$ and the value $\eta(0) = \eta_0 \in [0, h]$. These conditions and Eqs. (1.1) determine the distribution of the random vector $\{x(t), \eta(t)\}$ for $t > 0$ independently from the values $x(\tau)$ for $\tau < t - h$ and $\eta(\tau)$ for $\tau < t$.

Therefore in agreement with the concept considered in [3], in order to examine an element of the realized trajectory it is expedient to consider the segment $x(\omega, t + \vartheta)$ for $-h \leqslant \vartheta \leqslant 0$.

Thus the initial conditions $\{x_0(\vartheta_0), \eta_0\}$ and Eqs. (1.1) generate a random process which can be interpreted easily as a bunch of realizations of the motion, corresponding to all possible realizations of $\eta(\omega, t)$.

Whereupon it can be considered that each realization $\{x(\omega, t), \eta(\omega, t)\}$ of the process $\{x(t), \eta(t)\}$ satisfies Eq.

$$\lim_{\Delta t \to +0} \frac{x(\omega, t + \Delta t) - x(\omega, t)}{\Delta t} = Ax(\omega, t) + Bx(\omega, t - \eta(\omega, t))$$

The random process $x(t)$, determined in that manner will be denoted by the symbol $x(x_0(\vartheta_0), \eta_0, t)$, and the realizations of that process by the symbol $x(\omega, x_0(\vartheta_0), \eta_0, t)$.

2. For stochastic systems with a delay we may introduce definitions of the stability according to the probability, which generalize the definitions of Liapunov as it was done in [4] for the usual stochastic equations. Let us point out that the questions of the stability of usual stochastic equations were considered in [5 and 6] and others. The stability problem for systems with random delay has been considered by Lidskii [7].

Let us introduce for the completion of the exposition a few definitions analogous to the corresponding ideas of paper [7].

Let us denote by the symbol

$$\| x(\omega, x_0(\vartheta_0), \eta_0, t + \vartheta] \|^{(h)} = \\ = \sup \{ | x_i(\omega, x_0(\vartheta_0), \eta_0, t + \vartheta) |, \quad i = 1, \ldots, n, \; -h \leqslant \vartheta \leqslant 0 \}$$

Definition 2.1. The solution $x = 0$ of the system (1.1) is probably stable, if for any arbitrary small numbers $\varepsilon > 0$, $p > 0$ one may find a number $\delta > 0$, such that for any motion of the system (1.1) the following inequality is satisfied:

$$P\{[\sup_{t \geqslant 0} \| x(\omega, x_0(\vartheta_0), \eta_0, t + \vartheta) \|^{(h)}] < \varepsilon \mid \| x_0(\vartheta_0) \|^{(h)} \leqslant \delta\} > 1 - p \quad (2.1)$$

If, furthermore, for any number $\lambda > 0$, $q > 0$ and any arbitrary initial conditions $\{x_0(\vartheta_0), \eta_0\}$ one may find a number $T > 0$, such that

$$P\{[\sup_{t \geqslant T} \| x(\omega, x_0(\vartheta_0), \eta_0, t + \vartheta) \|^{(h)}] < \gamma\} > 1 - q \quad (2.2)$$

then the solution $x = 0$ is probably asymptotically stable.

For stochastic systems with a delay, we may develop a stability theory analogous to Liapunov's second method. Particularly the following statement is valid.

Theorem 2.1. If for Eqs. (1.1) there is a functional $W[x(\vartheta), \eta]$ positive definite for all $\eta \in [0, h]$ and having an infinitely small upper limit uniformly with respect to η and the quantity

$$\overline{\lim_{\Delta t \to +0}} \frac{\Delta M\{W\}}{\Delta t} = \\ = \overline{\lim_{\Delta t \to +0}} \frac{1}{\Delta t} [M\{W[x(x_0(\vartheta_0), \eta_0, \Delta t), \eta(\eta_0, \Delta t)] \mid x_0(\vartheta_0), \eta(0) = \eta_0\} - \\ - W[x_0(\vartheta_0), \eta_0]]$$

is negative definite, then the solution $x = 0$ of the system (1.1) is probably asymptotically stable.

Here $M\{\psi | B\}$ is the conditional mathematical expectation of the quantity ψ; the expression $\overline{\lim}\,(\Delta M\{W\}/\Delta t)$ represents the averaged higher derivative of the functional $W[x(\vartheta), \eta]$ on the basis of the system (1.1) and satisfies the inequality

$$\overline{\lim_{\Delta t \to +0}} \frac{\Delta M\{W\}}{\Delta t} \leqslant \overline{\lim} \left(\frac{\Delta W}{\Delta t}\right)_\eta + \int_0^h \{W[x(\vartheta), \beta] - W[x(\vartheta), \eta]\}\, d_\beta q(\eta, \beta) \quad (2.3)$$

where $\lim\,(\Delta W/\Delta t)_\eta$ is computed for a fixed value of η.

The proof of this statement is made along the same lines as the proof of the corresponding theorem for stochastic systems without delay [4] as long as the process has Markov properties in the space of the elements of the trajectory $\{x(\omega, t+\vartheta), \eta(\omega, t)\}$, $\vartheta \in [0, h]$. Let us note that Theorem (2.1) remains also valid when the functional $W[x(\vartheta), \eta]$ has *first order discontinuities in* η.

Together with the system (1.1) let us consider Eqs.

$$dx(t)/dt = Ax(t) + Bx(t - \xi) \quad (2.4)$$

where ξ is a constant delay, $\xi \in [0, h]$. We shall denote the solution of the system (2.4) for a fixed value of the delay ξ by the symbol

$$x(x_0(\vartheta_0), \xi, t + \vartheta)^\circ$$

Let for some value $\xi = \xi^\circ$, the undisturbed motion $x = 0$ of the system

$$dx(t)/dt = Ax(t) + Bx(t - \xi^\circ) \quad (2.5)$$

be stable, and assume that for any motion of Eqs. (2.5) there are constants $\alpha > 0$, $B_1 > 0$ such that the inequality

$$\| x(x_0(\vartheta_0), \xi^\circ, t + \vartheta)^\circ \|^{(h)} \leqslant B_1 \| x_0(\vartheta_0) \|^{(h)} e^{-\alpha t} \quad (2.6)$$

is valid.

It is then simple to show that for $t \geqslant t_0 \geqslant 2h$ the relation

$$\| x(x_0(t_0 + \vartheta_0), \xi^\circ, t + \vartheta)^\circ \|^{(2h)} \leqslant B_2 \| x_0(t_0 + \vartheta_0) \|^{(2h)} \exp(-\alpha(t - t_0))$$

where

$$\| x(t + \vartheta) \|^{(2h)} = \sup\{|x_i(t + \vartheta)|,\ i = 1, \ldots, n;\ -2h \leqslant \vartheta \leqslant 0\}$$

$$B_2 = B_1(1 + e^{-\alpha h})$$

is satisfied for the trajectories of the system (2.5).

It is known that for that condition we can construct a functional $V[x(\vartheta)]$ satisfying, on the trajectories of the system (2.5), the inequalities [9]

$$c_1 \| x(\vartheta) \|^{(2h)} \leqslant V[x(\vartheta)] \leqslant c_2 \| x(\vartheta) \|^{2h}$$

$$\overline{\lim_{\Delta t \to +0}} \left(\frac{\Delta V}{\Delta t}\right)_{\xi^\circ} \leqslant -c_3 \| x(\vartheta) \|^{(2h)} \quad (2.7)$$

$$| V[x''(\vartheta)] - V[x'(\vartheta)] | \leqslant c_4 \| x''(\vartheta) - x'(\vartheta) \|^{(2h)}$$

where c_1 to c_4 are positive constants. The functional $V[x(\vartheta)]$ can be chosen in the form

$$V[x_0(\vartheta_0)] = \int_{t_0}^{t_0+2h+T} \| x(x_0(\vartheta_0), \xi^\circ, \tau + \vartheta)^\circ \|^{(2h)} d\tau +$$

$$+ \sup\{ \| x\, x_0(\vartheta_0), \xi^\circ, \tau + \vartheta)^\circ \|^{(2h)} \quad t_0 \leqslant \tau \leqslant t_0 + 2h + T\} \quad (2.8)$$

$$t_0 \geqslant 2h, \qquad T = \frac{1}{\alpha} \ln 2B_2$$

3. We shall show sufficient conditions limiting the random delay $\eta(t)$ and for which the stability of the system (2.5) and the property (2.6) assure the asymptotic stability according

to the probability of the unperturbed motion of Eqs. (1.1). For that purpose we shall take some sufficiently small number $\gamma > 0$ and we shall consider the functional $W[x(\vartheta), \eta]$ determined by Eqs. [8]

$$W[x(\vartheta), \eta] = \begin{cases} V[x(\vartheta)] & \text{for } |\eta - \xi^\circ| < \gamma,\ \eta \in [0, h] \\ 2V[x(\vartheta)] & \text{for } |\eta - \xi^\circ| \geqslant \gamma,\ \eta \in [0, h] \end{cases} \tag{3.1}$$

The functional $W[x(\vartheta), \eta]$ is obviously positive definite for all $\eta \in [0, h]$ and accepts an infinitely small upper limit uniformly in η. We shall estimate the quantity $\overline{\lim}\ (\Delta M\{W\}/dt)$ along the trajectories of the system (1.1). Let us consider for $t_0 \geqslant 2h$, on the time interval $[t_0, t_0 + \Delta t]$, the trajectories of the systems (1.1) and (2.4) such that the initial curves $x_0(t_0 + \vartheta_0)$ coincide on the interval $[t_0 - 2h, t_0]$.

Then in agreement with the inequality (2.3) we have the estimates

$$\overline{\lim_{\Delta t \to +0}} \frac{\Delta M\{W\}}{\Delta t} \leqslant \overline{\lim_{\Delta t \to +0}} \left(\frac{\Delta V}{\Delta t}\right)_{\eta^\circ} + V[x_0(\vartheta_0)] \int\limits_{|\beta - \xi^\circ| \geqslant \gamma} d_\beta q(\eta^\circ, \beta) \tag{3.2}$$

for

$$\eta(t_0) = \eta^\circ,\ |\eta^\circ - \xi^\circ| < \gamma,\ \eta^\circ \in [0, h],\ x(t_0 + \vartheta_0) = x_0(\vartheta_0)$$

Here $\overline{\lim}\ (\Delta V/\Delta t)_{\eta^\circ}$ is computed on the basis of the system (2.4) for a fixed value $\xi = \eta^\circ$. Let us estimate this quantity. We have

$$\overline{\lim_{\Delta t \to +0}} \left(\frac{\Delta V}{\Delta t}\right)_{\eta^\circ} \leqslant \overline{\lim_{\Delta t \to +0}} \left(\frac{\Delta V}{\Delta t}\right)_{\xi^\circ} +$$

$$+ \overline{\lim_{\Delta t \to +0}} \frac{V[x(x_0(\vartheta_0), \eta^\circ, t_0 + \Delta t + \vartheta)^\circ] - V[x(x_0(\vartheta_0), \xi^\circ, t_0 + \Delta t + \vartheta)^\circ]}{\Delta t} \leqslant$$

$$\leqslant \overline{\lim_{\Delta t \to +0}} \left(\frac{\Delta V}{\Delta t}\right)_{\xi^\circ} + c_4 \| x(x_0(\vartheta_0), \eta^\circ, t_0 + \Delta t + \vartheta)^\circ -$$

$$- x(x_0(\vartheta_0), \xi^\circ, t_0 + \Delta t + \vartheta)^\circ \|^{(2h)} \tag{3.3}$$

For the system (2.4) we can get a constant $K > 0$ such, that the following inequality is valid:

$$\| x(x_0(\vartheta_0), \eta^\circ, t_0 + \Delta t + \vartheta)^\circ - x(x_0(\vartheta_0), \xi^\circ, t_0 + \Delta t + \vartheta)^\circ \|^{(2h)} \leqslant$$

$$\leqslant K \| x_0(\vartheta_0) \|^{(2h)} | \eta^\circ - \xi^\circ | < K\gamma \| x_0(\vartheta_0) \|^{(2h)} \tag{3.4}$$

Taking (2.7) and (3.4) into consideration, we get for the quantity (3.2) for $|\eta^\circ - \xi^\circ| < \gamma$, the following estimate:

$$\overline{\lim_{\Delta t \to +0}} \frac{\Delta M\{W\}}{\Delta t} \leqslant [-c_3 + K\gamma c_4 + c_2 Q_1(\eta^\circ)] \| x_0(\vartheta_0) \|^{(2h)}$$

$$\left(Q_1(\eta_0) = \int\limits_{|\beta - \xi^\circ| \geqslant \gamma} d_\beta q(\eta^\circ, \beta)\right) \tag{3.5}$$

By an analogous reasoning when $|\eta^\circ - \xi^\circ| \geqslant \gamma$ we can get the inequality

$$\overline{\lim_{\Delta t \to +0}} \frac{\Delta M\{W\}}{\Delta t} \leqslant [-2c_3 + 2Khc_4 - c_1 Q_2(\eta^\circ)] \| x_0(\vartheta_0) \|^{(2h)}$$

$$\left(Q_2(\eta^\circ) = \int\limits_{|\beta - \xi^\circ| < \gamma} d_\beta q(\eta^\circ, \beta)\right) \tag{3.6}$$

Now for an asymptotic stability according to the probability of the system (1.1) it is sufficient to require that the following conditions be satisfied:

$$\gamma < \frac{c_3}{2Kc_4}, \qquad Q_1(\eta^\circ) < \frac{c_3}{2c_2}, \qquad Q_2(\eta^\circ) > \frac{2(Khc_4 - c_3)}{c_1} \tag{3.7}$$

Thus the following statement is proven

Theorem 3.1. If the system (2.5) is stable, and the condition (2.6) is satisfied for some fixed value of the delay $\xi^\circ = \xi$, then we can find some positive constants N_1, N_2 such, that when the conditions

$$Q_1(\eta^\circ) < N_1 \quad \text{for } |\eta^\circ - \xi^\circ| < \gamma, \ \eta^\circ \in [0, h]$$

$$Q_2(\eta^\circ) > N_2 \quad \text{for } |\eta^\circ - \xi^\circ| \geqslant \gamma, \ \eta^\circ \in [0, h] \tag{3.8}$$

are satisfied, the motion $x = 0$ of the system (1.1) is probably asymptotically stable.

Note 3.1. The probability meaning of the quantities $Q_1(\eta^\circ)$ and $Q_2(\eta^\circ)$ is discerned from the equalities

$$P\{|\eta(t+\Delta t) - \xi^\circ| \geqslant \gamma \mid \eta(t) = \eta^\circ, |\eta^\circ - \xi^\circ| < \gamma\} = Q_1(\eta^\circ)\Delta t + o(\Delta t)$$

$$P\{|\eta(t+\Delta t) - \xi^\circ| < \gamma \mid \eta(t) = \eta^\circ, |\eta^\circ - \xi^\circ| \geqslant \gamma\} = Q_2(\eta^\circ)\Delta t + o(\Delta t)$$

which are consequences of (3.5) and (3.6). Consequently, the proven theorem means that random changes in the delay cannot affect a sufficiently strong system stability if the probability of a change from small values of the delay to large ones is small during the time Δt, and if the probability of the opposite changes is sufficiently large.

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